DIOPHANTINE ANALYSIS.

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62. Proposed by JOHN M. ARNOLD, Crompton, R. I.

Find, if possible, four square numbers in arithmetical progression.

I. Solution in Imaginaries by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

I have not, thus far, succeeded in obtaining a solution in real numbers, but the following in imaginaries.

Let $(x-y)^2$, x^2+y^2, $(x+y)^2$, $(x+y)^2+2xy$ be the numbers.

Let $x=m^2-n^2$, $y=2mn$.

$\therefore (m^2-n^2-2mn)^2$, $(m^2+n^2)^2$, $(m^2-n^2+2mn)^2$, $(m^2-n^2+2mn)^2+4mn(m^2-n^2)$.

The last is a square when $m=n$; let $m=n+1$.

$\therefore (1-2n^2)^2$, $(2n^2+2n+1)^2$, $(2n^2+4n+1)^2$, $4n^4+24n^3+32n^2+12n+1$.

Assume, $4n^4+24n^3+32n^2+12n+1=(2n^2+6n+k)^2$; also $4n^4+24n^3+(32+a^2)n^2+(12+2ab)n+1+b^2=(an+b)^2$.

$\therefore a^2=4+4k$, $6+ab=6k$, $k^2=1+b^2$.

$\therefore k^3-8k^2+17k-10=0$. $\therefore k=1$ or 2 or 5.

$\therefore n=0$ or $-\frac{1}{2}\pm\frac{1}{2}\sqrt{(-11)}$ or -1.

$\therefore 25\pm12\sqrt{(-11)}$, 25, $25\mp12\sqrt{(-11)}$, $25\mp24\sqrt{(-11)}$, or $[6\pm\sqrt{(-11)}]^2$, $(5)^2$, $[6\mp\sqrt{(-11)}]^2$, $\{\sqrt{[(25+\sqrt{6961})/2]}\mp\sqrt{[(25-\sqrt{6961})/2]}\}^2$.

II. Comment by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Let $x^2-2xy+y^2$, x^2+y^2, $x^2+2xy+y^2$, $x^2+4xy+y^2$ be the four numbers; two of them being squares, we have to make $x^2+y^2=\square$ ......(1), and $x^2+4xy+y^2=\square$ ......(2). Let $x=my$, then from (1) $m^2+1=\square=$ (say) $(p-m)^2$, from which $m=(p^2-1)/2p$. From (2), $m^2+4m+1=\square=$ (say) $(pm-1)^2$. $m=(2p+4)/(p^2-1)=(p^2-1)/2p$, or $(p^2-1)^2=4p^2+8p$. Hence $4p^2+8p=\square$. Or $p^2+2p=\square=$ (say) $q^2-2pq+p^2$, and $p=q^2/[2(q+1)]$. Then $4p^2+8p=[(q^2+2q)/(q+1)]^2$, and $p^2-1=(q^2+2q)/(q+1)$, and $p^2=(q^2+3q+1)/(q+1)$;
$$p=\sqrt{[(q^2+4q^2+4q+1)]}/(q+1).$$
The only methods, which I know, of making the numerator rational, give $q=0$, and $p=1$, and $m=0$.

Taking $p^2+2p=p^2q^2$ and proceeding in a similar manner as above, we get $p=[1/(q^2-1)]\sqrt{(q^4+4q^3-2q^2-4q+1)}$, and we get $q=0$, and $p=-1$ or $q=-1$ and $p=0$.

I have tried many other methods and all give $p=1$, or $p=-1$. While all this does not *demonstrate* that the question is impossible, I shall believe that it is so, until I see a solution.

63. Proposed by A. H. HOLMES, Brunswick, Me.

Given $x^3+y^3=20^3\times105489$, to find four positive *integral* values each for x and y.

Solution by the PROPOSER.

$x^3 + y^3 = 20^3 \times 105489$ or $= 843912000$. Take $x=1$, 2, etc., until we find $x=15$, $y=945$. Put $945=a$ and $15=b$. Then since there must be four values for each, we have, $x^3 + y^3 = a^3 + b^3$, or $x^3 - b^3 = a^3 - y^3$ ....(1).

Now suppose $x^3 + b^3 = a^3 - y^3$. Let $x = a - u$ and $b = mu - y$.

$\therefore a^3 - 3a^2u + 3au^2 - u^3 + m^3u^3 - 3m^3a^2u + 3muy^2 - y^3 = a^3 - y^3$.

Let $m = a^2/y^2$ and we have $u = 3ay^3/(a^3 + y^3)$.

$\therefore x = [a(a^3 - 2y^3)]/(a^3 + y^3)$ and $b = [y(2a^3 - y^3)]/(a^3 + y^3)$.

Now to find x and y in $x^3 - b^3 = a^3 - y^3$ it is evident one of the above values must be taken negatively, but it cannot be the value of b since the result would be at least one negative value.

$\therefore$ in (1) we have, $x = [y(2a^3 - y^3)]/(a^3 + y^3)$ and $b = [a(2y^3 - a^3)]/(a^3 + y^3)$. Whence we find $y = a\sqrt[3]{[(a+b)/(2a-b)]} = 945\sqrt[3]{(\frac{960}{1875})} = 945\sqrt[3]{(\frac{64}{125})}$.

$\therefore y = 756$, and we find $x = 744$.

$\therefore x = 15, 744, 756$, and 945 ; $y = 945, 756, 744$, and 15.

Also solved by *J. H. DRUMMOND*.

74. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

It is required to take from the proper *key* suitable material and hastily construct a "nest" of 10 or 15 prime, integral, rational trapeziums, each containing an area equal to the square root of the product of its four sides.

Solution by the PROPOSER.

If the business require great haste, write n, —two or more, convergents in the expansion of any number of the form of $\sqrt{(a^2+1)}$, say $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$, $\sqrt{17}$, $\sqrt{26}$, $\sqrt{37}$, etc. Observe the number of trapeziums, $[n(n-1)]/2$ is always triangular.

$\sqrt{17} = \frac{4}{1}$, $\frac{33}{8}$: $4^2 = \mathbf{16}$, $17 \times 1^2 = \mathbf{17}$, $17 \times 8^2 = \mathbf{1088}$, $33^2 = \mathbf{1089}$.

$\sqrt{10} = \frac{3}{1}$, $\frac{19}{6}$, $\frac{117}{37}$: $3^2 = 9$, $10 \times 1^2 = 10$, $19^2 = 361$, $10 \times 6^2 = 360$, $117^2 = 13689$, $10 \times 37^2 = 13690$. 9, 10, 360, 361 ; 9, 10, 13689, 13690 ; 360, 361, 13689, 13690.

$\sqrt{5} = \frac{2}{1}$, $\frac{9}{4}$, $\frac{38}{17}$, $\frac{161}{72}$: 4, 5, 80, 81; 4, 5, 1444, 1445 ; 4, 5, 25920, 25921 ; 80, 81, 1444, 1445 ; 80, 81, 25920, 25921 ; 1444, 1445, 25920, 25921.

$\sqrt{2} = \frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$: 1, 2, 8, 9 ; 1, 2, 49, 50 ; 1, 2, 288, 289 ; 1, 2, 1681, 1682 ; 8, 9, 49, 50 ; 8, 9, 288, 289 ; 8, 9, 1681, 1682 ; 49, 50, 288, 289 ; 49, 50, 1681, 1682 ; 288, 289, 1681, 1682.

Should it be desired to obtain several nests of this kind of rational trapeziums from a single series, take the convergents from the expansion of quantities of the form $\sqrt{(a^2+b)}$, where b is greater than 1.

$\sqrt{3} = \frac{1}{1}$, $\mathbf{2/1}$, $\frac{5}{3}$, $\mathbf{7/7}$, $\frac{19}{11}$, $\mathbf{26/15}$, $\frac{71}{41}$, $\mathbf{97/56}$, $\frac{265}{153}$: 3, 4, 48, 49 ; 3, 4, 675, 676 ; 3, 4, 9408, 9409 ; 48, 49, 675, 676 ; 48, 49, 9408, 9409 ; 675, 676, 9408, 9409 ; .....

1, 3, 25, 27 ; 1, 3, 361, 363 ; 1, 3, 5041, 5043 ; 1, 3, 70225, 70227 ; 27, 25, 361, 363 ; 27, 25, 5041, 5043 ; 27, 25, 70225, 70227 ; 361, 363, 5041, 5043 ; 361, 363, 70225, 70227 ; 5041, 5043, 70225, 70227.